Permutations avoiding a set of patterns $T \subseteq S_3$ and a pattern $\tau \in S_4$

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Abstract

In this paper we calculate the cardinality of the set $S_n(T,\tau)$ of all permutations in S_n that avoid one pattern from S_4 and a nonempty set of patterns from S_3.

The main body of the paper is divided into four sections corresponding to the cases $|T| = 1, 2, 3,$ and $|T| \geq 4$. At the end of each section we provide the tables accumulating all the results obtained.

1 Introduction

Let $[k] = \{1, \ldots, k\}$ be a (totally ordered) *alphabet* on k letters, and let $\alpha \in [k]^m$, $\beta \in [l]^m$ with $l \leq k$. We say that α is *order-isomorphic* to β if rhe following condition holds for all $1 \leq i < j \leq n$: $\alpha_i < \alpha_j$ if and only if $\beta_i < \beta_j$.

We say that $\tau \in S_n$ *avoids* $\alpha \in S_k$ (with $k \leq m$) if there is exist $1 \leq i_1 \leq \ldots \leq i_k \leq n$ such that $(\tau_{i_1}, \ldots, \tau_{i_k})$ is order-isomorphic to $\alpha = (\alpha_1, \ldots, \alpha_k)$, and we say that τ *avoids* α if τ does not contain α.

The set of all permutations in S_n avoiding α is denoted $S_n(\alpha)$. In a similar way, for any $A \subset \bigcup_{m=1}^{\infty} S_m$ we write $S_n(A)$ to denote the set of permutations in S_n avoiding all the permutations in A.

The study of the sets $S_n(\alpha)$ was initiated by Knuth [6], who peroved that $|S_n(\alpha)| = \frac{1}{n+1}\binom{2n}{n}$ for any $\alpha \in S_3$. Knuth's results where further extended in two directions. West [9] and Stankova [8] analyzed $S_n(\alpha)$ for

$\alpha \in S_4$ and obtained the complete classifiction, which contains 3 distinct cases. This classification, however does not give exact values of $S_n(\alpha)$. On the other hand, Simion and Schmidt [7] studied $S_n(T)$ for arbitrary subsets $T \subseteq S_3$ and discovered 7 distinct cases. The study of $S_n(\alpha, \tau)$ for all $\alpha \in S_3$, $\tau \in S_4$, τ avoids α, was completed by West [9], Billey, Jockusch and Stanley [1] and Guibert [5]. In the persent paper we continue this work and calculate the cardinalities of the sets $S_n(T, \alpha)$ for all nonempty sets $T \subseteq S_3$ and all permutations $\alpha \in S_4$. The rest of the intorduction contains several auxilary definitions and results.

We define two symmerties on permutations, the *reversal* $r : S_n \to S_n$ as follows:

$$r : (\alpha_1, \alpha_2, \ldots, \alpha_n) \longmapsto (\alpha_n, \ldots, \alpha_2, \alpha_1),$$

and the *inverse* $i : S_n \to S_n$ as follows:

$$i : \alpha \longmapsto \alpha^{-1}.$$

These symmetries can be extended to an arbitrary set of permutations $T \subset \bigcup_{m=1}^{\infty} S_m$ as follows:

$$\begin{aligned} r(T) &= \{r(\alpha) \mid \alpha \in T\}, \\ T^{-1} &= \{\alpha^{-1} \mid \alpha \in T\}. \end{aligned}$$

We denote by M_p the group of transformations of $\bigcup_{m=1}^{\infty} S_m$ generated by r and i, and we define *Symmetry classes* as the orbits of M_p in $\bigcup_{m=1}^{\infty} S_m$. In other words, A, B belongs to be same symmetry class $\mathcal{T}$ if there exist $g \in M_p$ such that $g(A) = B$.

Proposition 1.1 Let $A, B \subset \bigcup_{m=1}^{\infty} S_m$ belong to be same symmetry class $\mathcal{T}$. Then $|S_n(A)| = |S_n(B)|$.

Proof By Burstein [2] and defintions. ■

Let $b_1 < \ldots < b_n$; we denote by $S_{\{b_1,\ldots,b_n\}}$ the set of all permutation of the numbers $b_1, \ldots, b_n$; for example, $S_{\{1,\ldots,n\}}$ is just S_n. As above we denote by $S_{\{b_1,\ldots,b_n\}}(T)$ the set of all permutations in $S_{\{b_1,\ldots,b_n\}}$ avoiding all the permutations in T.

Proposition 1.2 Let $\tau \in S_{\{c_1,\ldots,c_k\}}$. Then there exists permutation $\alpha \in S_k$ such that $|S_n(\alpha)| = |S_{\{b_1,\ldots,b_n\}}(\tau)|$.

Proof We define a function $f : S_{\{c_1,\ldots,c_k\}} \to S_k$ by

$$f((c_{i_1}, c_{i_2}, \ldots, c_{i_k})) = (i_1, i_2, \ldots, i_k).$$

Then evidently $|S_{\{b_1,\ldots,b_n\}}(\tau)| = |S_n(f(\tau))|$. ■

Corollary 1.1 Let $T \subseteq S_{\{c_1,\ldots,c_k\}}$. Then there exists $R \subseteq S_k$ such that $|S_n(R)| = |S_{\{b_1,\ldots,b_n\}}(T)|$.

Proof Let $T = \{\tau_1, \ldots, \tau_l\}$ and $R = \{f(\tau_1), \ldots, f(\tau_l)\}$ where f is defined in proposition 1.2 . On the other hand by defintions

$$S_{\{b_1,\ldots,b_n\}}(T) = \bigcap_{i=1}^{l} S_{\{b_1,\ldots,b_n\}}(\tau_i).$$

Hence by the isomorphism f and proposition 1.2 this corollary holds. ■

2 $|S_n(\alpha, \tau)|$ for all $\alpha \in S_3$ and $\tau \in S_4$

Defintion 2.1 Let $\tau \in S_k$. We define A_τ^m as the set of all the words $\alpha = (\alpha_1, \ldots, \alpha_k)$ such that $1 \le \alpha_i \le m$ for all i and α is order-isomorphic to τ.

Proposition 2.1 $|A_\tau^m| = \binom{m}{k}$ for any $\tau \in S_k$ and $m \ge k$.

Proof If $m = k$ then the proposition is trivial. We proceed by induction. Let $|A_\tau^m| = \binom{m}{k}$, and we want to calculate the cardinality of the set A_τ^{m+1}. Assume that $\tau_d = k$. We have

$$A_\tau^{m+1} = \bigcup_{i=k}^{m+1} A^{i-1}_{\tau_1,\ldots,\tau_{d-1},\tau_{d+1},\ldots,\tau_k},$$

and the sets in above relation are disjoint. Hence by induction we obtain $|A_\tau^{m+1}| = \binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{m}{k-1} = \binom{m+1}{k}$. ■

Defintion 2.2 Let $\tau \in S_k$. We define V_τ^m as the set of all the permutations $\alpha \in S_m$ for which there exist $1 \le i_1 < i_2 < \ldots < i_k \le m$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k}) \in A_\tau^m$.

Example 2.1 $A^4_{132} = \{132, 142, 143, 243\}$.
$V^4_{132} = \{1324, 1342, 1432, 4132, 1423, 3142, 1243, 2143, 2431, 2413\}$, and by direct calculations we have $|V^4_\tau| = 10$ for all $\tau \in S_3$.

Defintion 2.3 Let $\nu : P(S_k) \to P(S_{k+1})$ be defined by $\nu(T) = \bigcup_{\tau \in T} V^{k+1}_\tau$ where $P(X)$ as usual stands for the power set of X; we denote $\nu^p(T) = \nu(\nu^{p-1}(T))$ for $p > 1$.

Lemma 2.1 $S_n(\nu(\tau)) = S_n(\tau)$ for all $\tau \in S_k$ and $n \geq k + 2$.

Proof Let $\alpha \in S_n(\tau)$ and assume that $\alpha \notin S_n(\nu(\tau))$, which means that α contains a subsequence $\beta \in \nu(\tau)$. So by defintions α contains a subsequence γ such that γ is order-isomorphic to τ, a contradiction. Hence $S_n(\tau) \subseteq S_n(\nu(\tau))$.

On the other hand, let α contain τ, which means that there exist $1 \leq i_1 < \ldots < i_k \leq n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ. Let $\alpha_{i_{m_1}}$ be the maximal element in $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$. If $\alpha_{m_1} \neq n$ then the subsequence $\alpha_{i_1}, \ldots, \alpha_{i_{d-1}}, n, \alpha_{i_{d+1}}, \ldots, \alpha_{i_k}$ of α is order-isomorphic to some permutation in the set V^{k+1}_τ, so $S_n(\nu(\tau)) \subseteq S_n(\tau)$. Hence we can assume that $\alpha_{m_1} = n$. Let α_{m_2} be the maximal element in $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\} \backslash \{\alpha_{m_1}\}$; by the same reason we see that the only nontrivial case is $\alpha_{m_2} = n - 1$ an so on. So $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is just a permutation of the numbers $n + 1 - k, \ldots, n$.

Now, since $n \geq k+2$, there exists d such that the subsequence $\alpha_{i_1}, \ldots, \alpha_{i_{d-1}}, 1, \alpha_{i_{d+1}}, \ldots, \alpha_{i_k}$ of α is order-isomorphic to some permutation in the set $\nu(\tau)$.

Hence in any case α contains some permutation β with $\beta \in \nu(\tau)$, which means that if α contains τ then $S_n(\nu(\tau)) \subseteq S_n(\tau)$. ■

Corollary 2.1 $S_n(\nu(T)) = S_n(T)$ for all $T \subseteq S_k$ and $n \geq k + 2$.

Proof By definitions $S_n(\nu(T)) = \bigcap_{\beta \in T} S_n(\nu(\beta))$, and by lemma 2.1 we obtain $S_n(\nu(T)) = \bigcap_{\beta \in T} S_n(\beta)$. Hence, again by definition, we obtain $S_n(\nu(T)) = S_n(T)$. ■

Theorem 2.1 $S_n(\nu^p(T)) = S_n(T)$ for all $T \subseteq S_k$ and $n \geq k + p + 1$.

Proof By definitions, corollary 2.1 and induction. ■

Theorem 2.2 Let $\alpha \in S_k$ and $\tau \in S_m$ with $k < m$. Then τ contains α if and only if $S_n(\alpha, \tau) = S_n(\alpha)$.

Proof Assume first that τ contains α. By defintions we have $S_n(\alpha,\tau) = S_n(\alpha) \cap S_n(\tau)$. On the other hand $\tau \in \nu^{m-k}(\alpha)$, and by theorem 2.1 we obtain $S_n(\alpha) = S_n(\nu^{m-k}(\alpha))$, which means that $S_n(\alpha,\tau) = S_n(\nu^{m-k}(\alpha))$, so again by theorem 2.1 we obtain $S_n(\alpha,\tau) = S_n(\alpha)$.

Now let $S_n(\alpha,\tau) = S_n(\alpha)$, then $S_n(\alpha) \subseteq S_n(\tau)$, which means that if $\beta \notin S_n(\tau)$ then $\beta \notin S_n(\alpha)$. By taking $\beta = \tau$, we get that τ must contain α. ■

Corollary 2.2 Let $\alpha \in S_3$, $\tau \in S_4$ and τ contain α. Then $|S_n(\alpha,\tau)| = c_n$ where c_n is the n-th Catalan number.

Proof By the theorem 2.2 and Knuth [6]. ■

Representative $T \in \mathcal{T}$	$\lvert\mathcal{T}\rvert$	*$\lvert S_n(T)\rvert$ for $T \in \mathcal{T}$*	*Reference*
$\{\alpha,\tau\}$ when $\alpha \in S_3$, $\tau \in S_4$ and τ contains α	60	$c_n = \frac{1}{n+1}\binom{2n}{n}$	corollary 2.2
$\overline{\{123,1432\}}$, $\overline{\{123,2143\}}$ $\overline{\{123,2413\}}$, $\overline{\{132,1234\}}$ $\overline{\{132,2134\}}$, $\overline{\{132,2314\}}$ $\overline{\{132,2341\}}$, $\overline{\{132,3241\}}$ $\overline{\{132,3412\}}$	46	f_{2n-2} where f_n is the n-th Fibonacci number	West [9]
$\overline{\{132,3421\}}$, $\overline{\{132,4231\}}$	12	$1+(n-1)2^{n-2}$	West [9] , Guibert [5] resp.
$\overline{\{123,2431\}}$	8	$3 \cdot 2^{n-1} - \binom{n+1}{2} - 1$	West [9]
$\overline{\{123,3421\}}$	4	$\binom{n}{4} + 2\binom{n}{3} + n$	West [9]
$\overline{\{132,3214\}}$	4	$\frac{(1-x)^3}{1-4x+5x^2-3x^3}$	West [9]
$\overline{\{132,4321\}}$	4	$\binom{n}{4} + \binom{n+1}{4} + \binom{n}{2} + 1$	West [9]
$\overline{\{123,4321\}}$	2	0	Erdös and Szekeres [3]
$\overline{\{123,3412\}}$	2	$2^{n+1} - \binom{n+1}{3} - 2n - 1$	Billey, Jockusch and Stanley [1]
$\overline{\{123,4231\}}$	2	$\binom{n}{5} + 2\binom{n}{4} + \binom{n}{3} + \binom{n}{2} + 1$	West [9]

Table 1: Cardinalities of the sets $S_n(\alpha,\tau)$ when $\alpha \in S_3$ and $\tau \in S_4$.

Billey, Jockusch and Stanley [1] show that $|S_n(123,3412)| = 2^{n+1} - \binom{n+1}{3} - 2n - 1$, Guibert [5] show that $|S_n(132,4231)| = 1 + (n-1)2^{n-2}$,

Erdös and Szekeres [3] show that $|S_n((1,\dots,l),(k,\dots,1))| = 0$. West [9] and corollary 2.2 compelete all the calculation of the cardinalities of $S_n(\alpha,\tau)$ when $\alpha \in S_3$ and $\tau \in S_4$. All these results we summarize in the above table.

The first column contains the list of representatives for the symmetry classes, one representative per each class. The second column contains the number of sets in each symmetry class, the third column contains expression for the cardinality of the sets in the corresponding symmetry class. The last column provides a reference to the paper (or theorem in the present papers) where this expression is obtained.

3 $|S_n(\alpha_1,\alpha_2,\tau)|$ when $\alpha_1 \neq \alpha_2 \in S_3$ and $\tau \in S_4$

In this section we calculate the cardinalities of all the sets $S_n(\alpha_1,\alpha_2,\tau)$ for any two different permutations $\alpha_1,\alpha_2 \in S_3$ and $\tau \in S_4$.

Proposition 3.1 Let α_1, α_2 be two different permutations in S_3 and let τ be permutation in S_4 such that τ contains α_1 or α_2. Then $|S_n(\alpha_1,\alpha_2,\tau)| = |S_n(\alpha_1,\alpha_2)|$.

Proof By theorem 2.2 and definitions. ∎

By proposition 3.1, Simion and Schmidt [7] and Erdös and Szekeres [3] we obtain the following theorem.

Theorem 3.1 For all $n \in \mathcal{N}$, $\tau \in S_4$:

1. $|S_n(\alpha_1,\alpha_2,\tau)| = 2^{n-1}$ if τ contains α_1 or α_2 and $(\alpha_1,\alpha_2) = (123,132)$, $(132,213)$, $(132,231)$, $(132,231)$ or $(132,312)$.
2. $|S_n(123,321,\tau)| = 0$ for all $n \geq 5$.
3. $|S_n(\alpha_1,\alpha_2,\tau)| = 0$ if $\alpha_1 \in S_3$, τ contains α_2, $n \geq 7$ and $\alpha_2 = 123$ or $\alpha_2 = 321$.

Let us analyse other cases.

Theorem 3.2 For all $n \in \mathcal{N}$,

$$|S_n(123,132,3214)| = |S_n(123,213,1432)| = |S_n(132,213,1234)| = t_n,$$

where t_n is the n-th Tribonacci number [4].

Proof 1. Let $\alpha \in G_n = S_n(123, 132, 3214)$, and let us consider the possible value of α_1:

1.1 $\alpha_1 \leq n-2$. Evidently there exist $\alpha_{i_j} = n+1-j$, $j = 1, 2$, hence α either contains $(\alpha_1, \alpha_{i_1}, \alpha_{i_2})$ which is order- isomorphic to 132, or contains $(\alpha_1, \alpha_{i_2}, \alpha_{i_1})$ which is order$-$isomorphic to 123, a contradiction.

1.2 $\alpha_1 = n-1$. Similarly to case 1.1 we have $\alpha_2 = n$ or $\alpha_2 = n-2$. If $\alpha_2 = n$ then $\alpha \in G_n$ if and only if $(\alpha_3, \ldots, \alpha_n) \in G_{n-2}$. If $\alpha_2 = n-2$ then $\alpha_3 = n$, since otherwise α contains 3214, hence $\alpha \in G_n$ if and only if $(\alpha_4, \ldots, \alpha_n) \in G_{n-3}$.

1.3 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + |G_{n-2}| + |G_{n-3}|$.

2. Let $\alpha \in G_n = S_n(123, 213, 1432)$, and let us consider the possible value of α_1:

2.1 $\alpha_1 \leq n-3$. Evidently there exist $\alpha_{i_j} = n+1-j$, $j = 1, 2, 3$. Since α avoids 123 we get $\alpha_{i_1} < \alpha_{i_2} < \alpha_{i_3}$, hence α contains $(\alpha_1, \alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})$ which is order$-$ isomorphic to 1432, a contradiction.

2.2 $\alpha_1 = n-2$. Let $\alpha_{i_j} = n+1-j$, $j = 1, 2$. Since α avoids 123 we get $i_1 < i_2$. If $i_1 \geq 3$ or $i_2 \geq 4$ then α contains 213, a contradiction. So $\alpha = (n-2, n, n-1, \alpha_4, \ldots, \alpha_n)$. Hence $\alpha \in G_n$ if and only if $(\alpha_4, \ldots, \alpha_n) \in G_{n-3}$.

2.3 $\alpha_1 = n-1$. If $\alpha_2 \leq n-2$ then α contains 213, so we have $\alpha = (n-1, n, \alpha_3, \ldots, \alpha_n)$. Hence $\alpha \in G_n$ if and only if $(\alpha_3, \ldots, \alpha_n) \in G_{n-2}$.

2.4 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + |G_{n-2}| + |G_{n-3}|$.

3. Let $\alpha \in G_n = S_n(132, 213, 1234)$, and let us consider the possible value of α_1:

3.1 $\alpha_1 \leq n-3$. Since α avoids 132 we get, similarly to case 2.1, that α contains $(\alpha_1, n-2, n-1, n)$, which is order$-$isomorphic to 1234, a contradiction.

3.2 $\alpha_1 = n-2$. If $\alpha_2 \leq n-2$ then α contains 213, and if $\alpha_2 = n$ then α contains 132, so we have that $\alpha_2 = n-1$. If $\alpha_3 \leq n-2$ then α contains 213, so $\alpha_3 = n$. Hence $\alpha = (n-2, n-1, n, \alpha_4, \dots, \alpha_n)$. That is, $\alpha \in G_n$ if and only if $(\alpha_4, \dots, \alpha_n) \in G_{n-3}$.

3.3 $\alpha_1 = n-1$. Similarly to case 2.3 we have $\alpha_2 = n$, so $\alpha = (n-1, n, \alpha_3, \dots, \alpha_n)$. Hence $\alpha \in G_n$ if and only if $(\alpha_3, \dots, \alpha_n) \in G_{n-2}$.

3.4 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \dots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + |G_{n-2}| + |G_{n-3}|$. ■

Theorem 3.3 For all $n \in \mathcal{N}$,

$$|S_n(123, 132, 3241)| = |S_n(132, 213, 2341)| = f_{n+2} - 1,$$

where f_n is the n-th Fibonacci number.

Proof 1. Let $\alpha \in G_n = S_n(123, 132, 3241)$ and let us consider the possible value of α_1:

1.1 $\alpha_1 \leq n-2$. Impossible, similarly to case 1.1 in theorem 3.2.

1.2 $\alpha_1 = n-1$. Similarly to case 1.2 in theorem 3.2 we have that $\alpha_2 = n-2$ or $\alpha_2 = n$. If $\alpha_2 = n-2$ then we get $\alpha_n = n$ since otherwise α contains 3241. Besides, $\alpha_i > \alpha_j$ for all $3 \leq i < j < n$ since otherwise α contains 123. So there is only one such permutation, namely $(n-1, n-2, \dots, 1, n)$. If $\alpha_2 = n$ then $\alpha \in G_n$ if and only if $(\alpha_3, \dots, \alpha_n) \in G_{n-2}$.

1.3 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \dots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + |G_{n-2}| + 1$. By the transformation $g_n = |G_n| + 1$ we get $g_n = g_{n-1} + g_{n-2}$. Besides $g_4 = 8$ and $g_5 = 13$, which means that $g_n = f_{n+2}$, hence $|G_n| = f_{n+2} - 1$ for all $n \geq 4$. It is easy to see that for $n = 1, 2, 3$ the same formula holds.

2. Let $\alpha \in K_n = S_n(132, 213, 2341)$, and let us consider the possible value of α_1:

2.1 $2 \leq \alpha_1 \leq n-2$. If $\alpha_2 = n$ then α contains 132. Let $\alpha_2 = n-1$, if $\alpha_3 = n$ then α contains 2341 otherwise α contains 213. Hence $\alpha_2 \leq n-2$. If $\alpha_2 < \alpha_1$ then α contains 213, so we have that $\alpha_2 > \alpha_1$. If $\alpha_3 > \alpha_2$ then α contains 2341, otherwise α contains 213, a contradiction.

2.2 $\alpha_1 = 1$. Since α avoids 132, we have only one permutation $(1, \ldots, n)$.

2.3 $\alpha_1 = n-1$. Since α avoids 213, we have that $\alpha_2 = n$. Hence $\alpha \in K_n$ if and only if $(\alpha_3, \ldots, \alpha_n) \in K_{n-2}$.

2.4 $\alpha_1 = n$. Evidently $\alpha \in K_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in K_{n-1}$.

Since the above cases are disjoint we obtain $|K_n| = |K_{n-1}| + |K_{n-2}| + 1$; besides $|K_4| = 7$ and $|K_5| = 12$. Similarly to the first part of the proof we obtain $|K_n| = |G_n| = f_{n+2} - 1$ for all $n \in \mathcal{N}$. ■

Theorem 3.4 For all $n \geq 3$,

$$|S_n(123, 132, 3421)| = |S_n(123, 213, 3421)| = 3n - 5.$$

Proof 1. Let $\alpha \in G_n = S_n(123, 132, 3421)$, and let us consider the possible value of α_1:

1.1 $\alpha_1 \leq n-2$. Impossible, similarly to case 1.1 in theorem 3.2.

1.2 $\alpha_1 = n-1$. Consider k such that $\alpha_k = n$. Since α avoids 3421 we have $\alpha_i < \alpha_j$ for all $k < i < j \leq n$, hence if $k \leq n-3$ then α contains 123. So either $k = n$ or $k = n-1$ or $k = n-2$. Besides $\alpha_i > \alpha_j$ for all $1 < i < j < k$ since otherwise α contains 123, and $\alpha_j > \alpha_i$ for $i < k < j$ since otherwise α contains 132. Hence in this case there are three possible permutations: $(n-1, \ldots, 1, n)$, $(n-1, \ldots, 2, n, 1)$ and $(n-1, \ldots, 3, n, 1, 2)$.

1.3 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + 3$. Besides $|G_3| = 4$, hence $|G_n| = 3n - 5$.

2. Let $\alpha \in K_n = S_n(123, 213, 3421)$, and let us consider the possible value of α_1:

2.1 $4 \leq \alpha_1 \leq n-1$. Let $\alpha_{i_j} = j$, $j = 1, 2, 3$, and $\alpha_{i_4} = n$. Since α avoids 213 we have that $i_4 < i_j$ for all $j = 1, 2, 3$. On the other hand α avoids 3421, so we have $i_1 < i_2 < i_3$, which means that α contains 123, a contradiction.

2.2 $\alpha_1 = n$. Evidently $\alpha \in K_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in K_{n-1}$.

2.3 $\alpha_1 = 1$. Since α avoids 123 we have only one permutation $(1, n, \dots, 2)$.

2.4 $\alpha_1 = 2$. Since α avoids 123 we have that α contains $(n, \dots, 3)$. Since α avoids 213 we have $\alpha_n = 1$. So we have only one permutation $(2, n, \dots, 3, 1)$.

2.5 $\alpha_1 = 3$. Since α avoids 123 and 213 we have two permutations $(3, n, \dots, 4, 1, 2)$ or $(3, n, \dots, 4, 2, 1)$, but α avoids 3421, so we we have only one permutation $(3, n, \dots, 4, 1, 2)$.

Since the above cases are disjoint we obtain $|K_n| = |K_{n-1}| + 3$. Besides $|K_3| = 4$, hence $|K_n| = |G_n| = 3n - 5$. ■

Theorem 3.5 Let $\alpha \in \{1432, 2143, 2431, 3214, 3241, 3421\}$, then for all $n \geq 2$,

$$|S_n(123, 312, \alpha)| = 2n - 2.$$

Proof 1. Let $\alpha \in G_n = S_n(123, 312, 1432)$. Fix k such that $\alpha_k = 1$ and let us consider the possible value of k:

1.1 $k = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_1, \dots, \alpha_{n-1}) \in S_{\{2,\dots,n\}}(123, 312, 1432)$.

1.2 $k = n - 1$. If $\alpha_n < n$ then α contains 312, so $\alpha_n = n$. On the other hand α avoids 123, so we have only one permutation $(n - 1, \dots, 1, n)$.

1.3 $k = n-2$. Similarly to the above case, we have $\alpha = (n-2, \dots, 1, n, n-1)$ or $\alpha = (n-2, \dots, 1, n-1, n)$, but the permutation $(n-2, \dots, 1, n-1, n)$ contains 123, so we have only one permutation $(n - 2, \dots, n, n - 1)$.

1.4 $1 \leq k \leq n - 3$. Since α avoids 123 we have that $\alpha_{n-2} > \alpha_{n-1} > \alpha_n$ and α contains 1432, a contradiction.

Since the above cases are disjoint we obtain by corollary 1.1 that $|G_n| = |G_{n-1}| + 2$. Besides $|G_3| = 4$, hence $|G_n| = 2n - 2$.

2. Let $\alpha \in G_n = S_n(123, 312, 2143)$. Fix k such that $\alpha_k = 1$ and let us consider the possible value of k:

2.1 $k = n$ or $k = n - 1$. Similarly to cases 1.1 or 1.2 resp.

2.2 $k = 1$. Since α avoids 123 we have only one permutation $(1, n, \dots, 2)$.

2.3 $2 \le k \le n-2$. Since α avoids 123 we have that $\alpha_{n-1} > \alpha_n$; since α avoids 312 we have $\alpha_n > \alpha_i$ and $\alpha_{n-1} > \alpha_i$ for all $i < k$, so α contains 2143, a contradiction.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + 2$. Besides $|G_3| = 4$, hence $|G_n| = 2n-2$.

3. Let $A = \{123, 312, 2431\}$ and $G_n = S_n(r(A^{-1})) = S_n(321, 132, 2314)$. By proposition 1.1 we get $|G_n| = |S_n(A)|$. Let $\alpha \in G_n$ and $\alpha_1 = t$. If $\alpha_2 \ge t+2$ then α contains 132, and if $2 \le \alpha_2 \le t-1$ then α contains 321, so we have $\alpha_2 = 1$ or $\alpha_2 = t+1$.

Let $\alpha_2 = 1$; since α avoids 132 we have that $\alpha = (t, 1, \dots, t-1, t+1, \dots, n)$, anthere are $n-1$ permutations of this type.

Let $\alpha_2 = t+1$; since α avoids 132 we have that α conatins $(t, t+1, \dots, n)$. If $\alpha_i < t$ and $3 \le i \le n-t+1$ then α contains 2314, which means that $\alpha = (t, \dots, n, \alpha_{n-t+2}, \dots, \alpha_n)$. Since α avoids 321 we have that $\alpha = (t, \dots, n, 1, \dots, t-1)$, and there are $n-1$ permutations of this type. Hence $|G_n| = 2n-2$.

4. Let $\alpha \in G_n = S_n(123, 312, 3214)$ and $\alpha_1 = t$. If $t+1 \le \alpha_2 \le n-1$ then α contains 123, and if $\alpha_2 \le t-2$ then α contains 312, so we have that $\alpha_2 = t-1$ or $\alpha_2 = n$.

Let $\alpha_2 = n$; since α avoids 312 we get $\alpha = (t, n, \dots, t+1, t-1, \dots, 1)$, and there are $n-1$ permutations of this type.

Let $\alpha_2 = t-1$; if $t+1 \le \alpha_3 \le n-1$ then α contains 123, and if $\alpha_3 \le t-2$ then α contains 3214, so we have that $\alpha_3 = n$. Since α avoids 312 we have that $\alpha = (t, t-1, n, \dots, t+1, t-2, \dots, 1)$, and there are $n-1$ permutations of this type. Hence $|G_n| = 2n-2$.

5. Let $\alpha \in G_n = S_n(123, 312, 3241)$ and $\alpha_1 = t$. Similarly to case 4 we have that $\alpha_2 = t-1$ or $\alpha_2 = n$.

Let $\alpha_2 = n$; since α avoids 312 we have that $\alpha = (t, n, \dots, t+1, t-1, \dots, 1)$, and there are $n-1$ permutations of this type.

Let $\alpha_2 = t-1$. If $t=2$ then, since α avoids 123 we get $\alpha = (2,1,n,\dots,3)$. If $t \geq 3$ then, since α avoids 312 we have that α contains $(t-1,t-2,\dots,1)$. If there exist $3 \leq i < t$ such that $\alpha_i > t$, then α contains 3241, so $\alpha = (t,\dots,1,\alpha_{t+1},\dots,\alpha_n)$. Since α avoids 123 we have that $\alpha = (t,\dots,1,n,\dots,t+1)$, and there are $n-2$ permutations of this type. Hence $|G_n| = 2n-2$.

6. Let $\alpha \in G_n = S_n(123,312,3421)$, fix k such that $\alpha_k = 1$ and let us consider the possible value of k:

6.1 $k \leq n-1$. Since α avoids 312 we have that $\alpha_i < \alpha_j$ for all $i<k<j$, so since α avoids 123 we get $\alpha_i > \alpha_j$ for all $i<j<k$ or $k<i<j$. Hence $\alpha = (k,k-1,\dots,1,n,n-1,\dots,k+1)$, and there are $n-1$ permutations of this type.

6.2 $k=n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_1,\dots,\alpha_{n-1}) \in S_{n-1}(123,312,231)$. By Simion and Schmidt [7] we have $|S_{n-1}(123,312,231)| = n-1$.

Since the above cases are disjoint we obtain $|G_n| = n-1+n-1 = 2n-2$.
■

Theorem 3.6 For all $n \in \mathcal{N}$,

$$|S_n(\alpha_1,\alpha_2,\tau)| = \binom{n}{2} + 1,$$

in the following cases:

1. $\alpha_1 = 123$, $\alpha_2 = 231$ and $\tau \in S_4$ contains 123 or 231.
2. $\alpha_1 = 123$, $\alpha_2 \in \{132, 213\}$ and $\tau \in \{3412, 4231\}$.
3. $(\alpha_1,\alpha_2,\tau) = (132,213,3412)$.
4. $\alpha_1 = 132$, $\alpha_2 = 231$ and $\tau \in \{1234, 2134, 3124, 3214\}$.
5. $(\alpha_1,\alpha_2,\tau) = (213,312,3412)$.
6. $\alpha_1 = 213$, $\alpha_2 = 321$ and $\tau \in \{1324, 2314, 1324\}$.
7. $(\alpha_1,\alpha_2,\tau) = (213,132,4321)$.

Proof 1. By Simion and Schmidt [7] we get $|S_n(123,231)| = \binom{n}{2}+1$, hence by theorem 2.2 we have that $|S_n(123,231,\tau)| = \binom{n}{2}+1$ for all $\tau \in S_4$ containing 123 or 231.

2. Let $\alpha \in G_n = S_n(123,132,3412)$ and let us consider the possible value of α_1:

2.1 $\alpha_1 \leq n-2$. Impossible, similarly to case 1.1 in theorem 3.4

2.2 $\alpha_1 = n-1$. Let $\alpha_k = n$; since α avoids 123 we have for all $i_1 < i_2 < k$, $\alpha_{i_1} > \alpha_{i_2}$. Since α avoids 3412 we have for all $k < i_1 < i_2$, $\alpha_{i_1} > \alpha_{i_2}$. Since α avoids 132 we have $\alpha = (n-1,\dots,n-k+1,n,n-k,\dots,1)$, and there are $n-1$ permutations of this type.

2.3 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2,\dots,\alpha_n) \in G_{n-1}$.

Since the above casses are disjoint we obtain $|G_n| = |G_{n-1}|+n-1$. Besides $|G_4| = 7$, hence $|G_n| = \binom{n}{2}+1$ for all $n \geq 4$. It is easy to see that for $n = 1,2,3$ the same formula holds for all $n \in \mathcal{N}$.

3. Let $\alpha \in G_n = S_n(123,132,4231)$ and let us consider the possible value of α_1:

3.1 $\alpha_1 \leq n-2$. Impossible, similarly to case 1.1 in theorem 3.4.

3.2 $\alpha_1 = n-1$. Consider k such that $\alpha_k = n$ and $k \leq n-1$. Since α avoids 123 we get $\alpha_i > \alpha_j$ for all $i < j < k$, and since α avoids 132 we get $\alpha_i > \alpha_j$ for all $i < k < j$, So $\alpha = (n-1,n-2,\dots,n-k+1,n,\alpha_{k+1},\dots,\alpha_n)$. Hence $\alpha \in G_n$ if and only if $(\alpha_{k+1},\dots,\alpha_n) \in S_{n-k}(123,132,231)$. By Simion and Schmidt [7] we get $|S_{n-k}(123,132,231)| = n-k$.

If $k = n$ then since α avoids 123 we have only one permutation $(n-1,\dots,1,n)$.

3.3 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2,\dots,\alpha_n) \in S_{n-1}(123,132,231)$. By Simion and Schmidt [7] we have $|S_{n-1}(123,132,231)| = n-1$.

Since the above cases are disjoint we obtain $|G_n| = 1+(1+\ldots+n-2)+n-1 = \binom{n}{2}+1$.

4. Let $\alpha \in G_n = S_n(123,213,3412)$. Let us consider the possible value of α_1:

4.1 $\alpha_1 = 1$. Since α avoids 123 we have that $\alpha = (1, n, \ldots, 2)$.

4.2 $\alpha_1 = t$, $2 \leq t \leq n-1$. If $\alpha_2 \leq t-1$ then α contains 213, and if $t+1 \leq \alpha_2 \leq n-1$ then α contains 123, So $\alpha_2 = n$. Since α avoids 3412 we have that α contains $(t-1, \ldots, 1)$.

Fix k such that $\alpha_k = t-1$. If there $i > k$ such that $\alpha_i > t$ then α contains $(t, t-1, \alpha_i)$ which is order-isomorphic to 213, so $\alpha = (t, n, \alpha_3, \ldots, \alpha_{n-t+1}, t-1, \ldots, 1)$. Since α avoids 123 we have that $\alpha = (t, n, \ldots, t+1, t-1, \ldots, 1)$, and there are $n-2$ permutations of this type.

4.2 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + n - 1$. Besides $|G_4| = 7$, hence similarly to the second proof we get $|G_n| = \binom{n}{2} + 1$.

5. Let $A = \{123, 213, 4231\}$ and $G_n = S_n(r(A)) = S_n(321, 312, 1324)$. By proposition 1.1 we get $|G_n| = |S_n(A)|$. Let $\alpha \in G_n$ and let us consider the possible value of α_1:

5.1 $3 \leq \alpha_1$. Since α avoids 312 we have that α contains 321, a contradiction.

5.2 $\alpha_1 = 2$. Consider k such that $\alpha_k = 1$. Since α avoids 321 we have $\alpha_i < \alpha_j$ for all $2 \leq i < j \leq k-1$, and since α avoids 312 we have that $\alpha_i < \alpha_j$ for all $i < k < j$. Therefore $\alpha = (2, \ldots, k, 1, \alpha_{k+1}, \ldots, \alpha_n)$. Hence $\alpha \in G_n$ if and only if $(\alpha_{k+1}, \ldots, \alpha_n) \in S_{\{k+1,\ldots,n\}}(321, 312, 213)$. By Simion and Schmidt [7] and corollary 1.3 we get that $S_{\{k+1,\ldots,n\}}(321, 312, 213)$ contains $n-k$ permutations for all $k \leq n-1$, and exactly one permutation for $k = n$.

5.3 $\alpha_1 = 1$. Similarly to case 5.2, $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in S_{\{2,\ldots,n\}}(321, 312, 213)$. By Simion and Schmidt [7] and corollary 1.1 we have that $n-1$ permutations.

Since the above cases are disjoint we obtain $|G_n| = (n-2+\ldots+1)+1+n-1 = \binom{n}{2} + 1$.

6. Let $\alpha \in G_n = S_n(132, 213, 3412)$ and let us consider the possible value of α_1:

6.1 $\alpha_1 = t \leq n-1$. Fix k such that $\alpha_k = n$. If $\alpha_i < t$ and $i < k$ then α contains 213, and if $\alpha_i > t$ and $i > k$ then α contains 132. Since α avoids 132 we have that $\alpha = (t, t+1, \ldots, n, \alpha_{n-t+2}, \ldots . \alpha_n)$, and since α avoids 3412 we get $\alpha = (t, t+1, \ldots, n, t-1, \ldots, 1)$, and there are $n-1$ permutations of this type.

6.2 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + n - 1$. Besides $|G_4| = 7$, hence similarly to the second proof we get $|G_n| = \binom{n}{2} + 1$.

7. Let $\alpha \in G_n = S_n(132, 231, 1234)$ and let us consider the possible value of α_1:

7.1 $\alpha_1 = 1$. Since α avoids 132 we get $\alpha = (1, \ldots, n)$, so α contains 1234, a contradiction.

7.2 $2 \leq \alpha_1 \leq n-3$. Since α avoids 132 the permutation α contains $(\alpha_1, n-2, n-1, n)$ which is order-isomorphic to 1234, a contradiction.

7.3 $\alpha_1 = n-2$. Since α avoids 231 we have that $\alpha_n, \alpha_{n-1} \in \{n-1, n\}$, and since α avoids 132 we get $\alpha_{n-1} = n-1$ and $\alpha_n = n$. Since α avoids 1234 we have that $\alpha_i > \alpha_j$ for all $2 \leq i < j \leq n-2$. Hence $\alpha = (n-2, n-3, \ldots, 1, n-1, n)$.

7.4 $\alpha_1 = n-1$. If $\alpha_n \leq n-2$ then α contains 231, so $\alpha_n = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_{n-1}) \in S_{n-2}(132, 231, 123)$. By Simion and Schmidt [7] we have $|S_{n-2}(132, 231, 123)| = n-2$.

7.5 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + n - 2 + 1$. Besides $|G_4| = 7$, hence similarly to the second proof we get $|G_n| = \binom{n}{2} + 1$.

8. Let $\alpha \in G_n = S_n(132, 231, 2134)$ and let us consider the possible value of α_1:

8.1 $\alpha_1 = 1$. Since α avoids 132 we get $\alpha = (1, 2, \ldots, n)$.

8.2 $2 \leq \alpha_1 \leq n-2$. Since α avoids 132 we get that α contains $(\alpha_1, n-1, n)$ and since α avoids 231 we get that α contains $(\alpha_1, 1, n-1, n)$ which is order-isomorphic to 2134, a contradiction.

8.3 $\alpha_1 = n-1$. Similarly to case 7.4 we get $\alpha_n = n$. So $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_{n-1}) \in S_{n-2}(132, 231, 213)$. By Simion and Schmidt [7] we have $|S_{n-2}(132, 231, 213)| = n-2$.

8.4 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + n - 2 + 1$. Besides $|G_4| = 7$, hence similarly to the second proof we get $|G_n| = \binom{n}{2} + 1$.

9. Let $\alpha \in G_n = S_n(132, 231, 3124)$ and let us consider the possible value of α_1:

9.1 $\alpha_1 = 1$. Similarly to case 8.1 we have only one permutation $(1, 2, \ldots, n)$.

9.2 $\alpha_1 = 2$. Since α avoids 231 we have that $\alpha_2 = 1$, and since α avoids 132 we get $\alpha = (2, 1, 3, \ldots, n)$.

9.3 $\alpha_1 = t$, $3 \leq t \leq n-1$. Fix k such that $\alpha_k = 1$. Since α avoids 231 we have that $\alpha_i > \alpha_j$ for all $i < j \leq k$ and $\alpha_i \leq t$ for all $i \leq k$. Since α avoids 132 we have that $\alpha_i < \alpha_j$ for all $k < i < j$. If $\alpha_i < t$ and $i > k$ then α contains either $(t, 1, \alpha_i, n)$ or $(t, 1, n, \alpha_i)$, which means that α contains 3124 or 132, a contradiction. Hence $\alpha = (t, t-1, \ldots, 1, t+1, t+2, \ldots, n)$, and there are $n-3$ permutations of this type.

9.4 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + n - 3 + 1 + 1$. Besides $|G_4| = 7$, hence similarly to the second proof we get $|G_n| = \binom{n}{2} + 1$.

10. Let $\alpha \in G_n = S_n(132, 231, 3214)$ and let us consider the possible value of α_1:

10.1 $\alpha_1 = 1$. Similarly to case 8.1 we have only one permutation $(1, 2, \ldots, n)$.

10.2 $\alpha_1 = t$, $2 \leq t \leq n-1$. Since α avoids 132 we have that α contains $(t, t+1, \ldots, n)$. Fix k such that $\alpha_k = t+1$, if $\alpha_i < t$ and $i > k$ then α contains 231. Hence $\alpha = (t, \alpha_2, \ldots, \alpha_t, t+1, \ldots, n)$. Since α avoids 3214 we have that $\alpha = (t, 1, \ldots, t-1, t+1, \ldots, n)$, and there are $n-2$ permutations of this type.

10.3 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in G_{n-1}$.

Since th above cases are disjoint we obtain $|G_n| = |G_{n-1}| + n - 1$. Besides $|G_4| = 7$, hence similarly to the second proof we get $|G_n| = \binom{n}{2} + 1$.

11. Let $\alpha \in G_n = S_n(213, 312, 2341)$ and let us consider the possible value of α_1:

11.1 $\alpha_1 = 1$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in S_{\{2,\ldots,n\}}(213, 312, 2341)$.

11.2 $\alpha_1 = t$, $2 \leq t \leq n-1$. If $\alpha_2 \leq t-1$ then α contains 213, and if $t+1 \leq \alpha_2 \leq n-1$ then α contains $(t, \alpha_2, 1, n)$ or $(t, \alpha_2, n, 1)$ which is order-isomorphic to 213 or 2341 respectively, so $\alpha_2 = n$. Since α avoids 312 we have that $\alpha = (t, n, n-1, \ldots, t+1, t-1, \ldots, 1)$, and there are $n-2$ permutations of this type.

11.2 $\alpha_1 = n$. Since α avoids 312 we have that $\alpha = (n, \ldots, 1)$.

Since the above cases are disjoint we obtain by corollary 1.1 that $|G_n| = |G_{n-1}| + n - 2 + 1$. Besides $|G_4| = 7$, hence similarly to the second proof we get $|G_n| = \binom{n}{2} + 1$.

12. By Simion and Schmidt [7] we have that $|S_n(213, 321)| = \binom{n}{2} + 1$, hence by theorem 2.9 we get $|S_n(213, 321, \tau)| = \binom{n}{2} + 1$ when $\tau \in \{1324, 2314\}$.

Proof(13): Let $A = \{231, 312, 1324\}$ and $G_n = S_n(r(A)) = S_n(132, 213, 4231)$. By proposition 1.1 we get $|G_n| = |S_n(A)|$. Let $\alpha \in G_n$ and let us consider the possible value of α_1:

13.1 $\alpha_1 = 1$. Since α avoids 132 we have that $\alpha = (1, 2, \ldots, n)$.

13.2 $\alpha_1 = t$, $2 \leq t \leq n-1$. Similarly to case 6.1 we have that $\alpha = (t, t+1, \ldots, n, \alpha_{n-t+2}, \ldots, \alpha_n)$. Evidently $\alpha \in G_n$ if and only if $(\alpha_{n-t+2}, \ldots, \alpha_n) \in S_{t-1}(132, 213, 231)$. By Simion and Schmidt [7] we have $|S_{t-1}(132, 213, 231)| = t-1$.

13.3 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in S_{n-1}(132, 213, 231)$. By Simion and Schmidt [7] we have that $|S_{n-1}(132, 213, 231)| = n-1$.

Since the above cases are disjoint we obtain $|G_n| = 1 + (1 + 2 + \ldots + n - 2) + n - 1 = \binom{n}{2} + 1$.

14. Let $A = \{213, 132, 4321\}$ and $G_n = S_n(r(A)) = S_n(132, 213, 4231)$. By proposition 1.1 we get $|G_n| = |S_n(A)|$. Let $\alpha \in G_n$ and let us consider the possible value of α_1:

14.1 $\alpha_1 = 1$. Since α avoids 132 we have that $\alpha = (1, 2, \ldots, n)$.

14.2 $\alpha_1 = t$, $2 \leq t \leq n-2$. Fix k such that $\alpha_k = n$, since α avoids 213 and 132 we have that $\alpha_i \geq t$ for $i \leq k$ and $\alpha_i \leq t-1$ for $i \geq k+1$ respectively, so $\alpha = (t, t+1, \ldots, n, \alpha_{n-t+2}, \ldots, \alpha_n)$. Evidently $\alpha \in G_n$ if and only if $(\alpha_{n-t+2}, \ldots, \alpha_n) \in S_{t-1}(213, 132, 321)$. By Simion and Schmidt [7] we have $|S_{t-1}(213, 132, 321)| = t-1$.

14.3 $\alpha_1 = n-1$. Since α avoids 213 we have that $\alpha_2 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_3, \ldots, \alpha_n) \in S_{n-2}(213, 132, 321)$. By Simion and Schmidt [7] we have that $|S_{n-2}(213, 132, 321)| = n-2$.

14.4 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in S_{n-1}(213, 132, 321)$. By Simion and Schmidt [7] we have that $|S_{n-1}(213, 132, 321)| = n-1$.

Since the above cases are disjoint we obtain $|G_n| = 1 + (1 + 2 + \ldots + n - 3) + n - 2 + n - 1 = \binom{n}{2} + 1$. ■

All these results we summarize in the table 2 (next page).

4 $|S_n(T, \tau)|$ when $T \subset S_3$, $|T| = 3$ and $\tau \in S_4$

In this section we calculate the cardinalities of all the sets $S_n(T, \tau)$ when $|T| = 3$, $T \subset S_3$ and τ is any permutation in S_4.

Theorem 4.1 Let $T = \{123, 132, 213\}$ and let $\tau \in S_4$ contain at least one permutation in T. Then $|S_n(T, \tau)| = |S_n(T)| = f_{n+1}$, where f_n is the n-th Fibonacci number.

Proof By Simion and Schmidt [7] we have that $|S_n(T)| = f_{n+1}$, hence by theorem 2.2 we get $|S_n(T, \tau)| = f_{n+1}$. ■

Theorem 4.2 Let $T \subset S_3$ and $|T| = 3$. For all $n \geq 6$,

$$|S_n(T, \tau)| = 0,$$

in the following cases:

Representative $T \in \mathcal{T}$	$\lvert\mathcal{T}\rvert$	*$\lvert S_n(T)\rvert$ for $T \in \mathcal{T}$*	*Reference*
$\{\alpha_1, \alpha_2, \tau\}$ when $\tau \in S_4$, τ contains α and $\overline{\{\alpha_1, \alpha_2\}} = \overline{\{123, 132\}}$,	160	2^{n-1}	theorem 3.1
$\overline{\{123, 132, 3412\}}$, $\overline{\{123, 132, 4231\}}$ $\overline{\{123, 213, 3412\}}$, $\overline{\{123, 213, 4231\}}$ $\overline{\{132, 213, 3412\}}$, $\overline{\{132, 231, 1234\}}$ $\overline{\{132, 231, 2134\}}$, $\overline{\{132, 231, 3124\}}$ $\overline{\{132, 231, 3214\}}$, $\overline{\{213, 312, 2341\}}$ $\overline{\{213, 312, 1324\}}$, $\overline{\{213, 321, 2314\}}$ $\overline{\{231, 312, 1324\}}$, $\overline{\{132, 213, 4321\}}$ $\{123, 231, \tau\}$ when $\tau \in S_4$ contains 123 or 231	118	$\binom{n}{2} + 1$	theorem 3.6
$\{123, 321, \tau\}$, $\tau \in S_4$ $\{123, \alpha, 4321\}$, $\alpha \in S_3$, $\alpha \neq 123$ $\{321, \alpha, 1234\}$, $\alpha \in S_3$, $\alpha \neq 321$	32	0	Erdös and Szekeres [3]
$\overline{\{123, 312, \tau\}}$, when $\tau = 1432$, $\tau = 2143$, 2431, 3214, 3241 or 3421	24	$2n - 2$	theorem 3.5
$\overline{\{123, 132, 3241\}}$, $\overline{\{132, 213, 2341\}}$	12	$f_{n+2} - 1$	theorem 3.3
$\overline{\{123, 132, 3421\}}$, $\overline{\{123, 213, 3421\}}$	8	$3n - 5$	theorem 3.4
$\overline{\{123, 132, 3214\}}$, $\overline{\{123, 213, 1432\}}$ $\overline{\{132, 213, 1234\}}$	6	a_n, where a_n is the n-th Tribonacci number [4]	theorem 3.2

Table 2: Cardinalities of the sets $S_n(\alpha_1, \alpha_2, \tau)$ when $\alpha_1, \alpha_2 \in S_3$ and $\tau \in S_4$.

1. $123 \in T$ and $\tau = 4321$.
2. $321 \in T$ and $\tau = 1234$.
3. $\{123, 321\} \subset T$ and $\tau \in S_4$.

Proof By Erdös and Szekeres [3]. ■

Theorem 4.3 For all $n \in \mathcal{N}$,

$$|S_n(T, \tau)| = n,$$

in the following cases:

1. $T = \{123, 132, 231\}$ and $\tau \in S_4$ contains at least one permutation in T.

2. $T = \{123, 132, 213\}$ and $\tau = 3412$.

Proof 1. By Simion and Schmidt [7] we have that $|S_n(T)| = n$, hence by theorem 2.2 we get $|S_n(T, \tau)| = n$ when $\tau \in S_4$ contains a permutation in T.

2. Let $\alpha \in G_n = S_n(123, 132, 213, 3412)$ and let us consider the possible value of α_1:

2.1 $\alpha_1 \leq n-2$. Since α avoids 123 we have that α contains $(\alpha_1, n, n-1)$, which means that α contains 132, a contradiction.

2.2 $\alpha_1 = n-1$. Since α avoids 213 we get $\alpha_2 = n$. Since α avoids 3412 we have only one permutation $(n-1, n, n-2, \ldots, 1)$.

2.3 $\alpha_1 = n$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2, \ldots, \alpha_n) \in G_{n-1}$.

Since the above cases are disjoint we obtain $|G_n| = |G_{n-1}| + 1$. Besides $|G_4| = 4$, hence $|G_n| = n$ for all $n \geq 4$. It is easy to see that for $n = 1, 2, 3$ the same formula holds. ■

Proposition 4.1 Let $T \subseteq S_k$. If $\alpha \notin S_n(T)$ then $(\alpha_1, \ldots, \alpha_{j-1}, n+1, \alpha_j, \ldots, \alpha_n) \notin S_{n+1}(T)$ for all $1 \leq j \leq n$.

Proof By definitions. ■

Theorem 4.4 For all $n \geq 4$,

1. $S_n(123, 132, 213, 3421) = \{(n-1, n, n-2, \ldots, 1),\ (n-1, n, n-1, \ldots, 1, 2),\ (n, \ldots, 3, 1, 2),\ (n, \ldots, 1)\}$.
2. $S_n(123, 132, 213, 4231) = \{(n, \ldots, 5, 3, 4, 1, 2), (n, \ldots, 4, 2, 3, 1), (n, \ldots, 3, 1, 2), (n, \ldots, 1)\}$.

Proof By induction and proposition 4.1 . ■

Theorem 4.5 $\delta_n = (1, 2, \ldots, n)$. For all $3 \leq n$,

1. $S_n(123, 132, 231, 3214) = \{(n, \ldots, 4, 2, 1, 3),\ (n, \ldots, 3, 1, 2),\ \delta_n\}$.
2. $S_n(123, 132, 231, 4312) = \{(n-1, \ldots, 1, n),\ (n, n-2, \ldots, 1, n-1),\ \delta_n\}$.
3. $S_n(123, 132, 231, 4213) = \{(n-1, \ldots, 1, n),\ (n, \ldots, 3, 1, 2),\ \delta_n\}$.

4. $S_n(123, 231, 312, 1432) = \{(n-2, \ldots, 1, n, n-1), \ (n-1, \ldots, 1, n), \ \delta_n\}$.
5. $S_n(123, 231, 312, 2143) = \{(2, 1, n, \ldots, 3), \ (n-1, \ldots, 1, n), \ \delta_n\}$.
6. $S_n(132, 213, 231, 1234) = \{(n, \ldots, 4, 1, 2, 3), \ (n, \ldots, 3, 1, 2), \ \delta_n\}$.
7. $S_n(132, 213, 231, 4123) = \{r(\delta_n), \ (n, \ldots, 3, 1, 2), \ \delta_n\}$.
8. $S_n(132, 213, 231, 4312) = \{r(\delta_n), \ (n, 1, \ldots, n-1), \ \delta_n\}$.
9. $S_n(132, 213, 231, 4321) = \{r(\delta_n), \ (n, 1, \ldots, n-1), \ (n, n-1, 1, \ldots, n-2)\}$.

Proof By induction and Proposition 4.1. ■

All these results we summarize in the following table.

Representative $T \in \mathcal{T}$	$\lvert\mathcal{T}\rvert$	$\lvert S_n(T)\rvert$ *for* $T \in \mathcal{T}$	*Reference*
$T \cup \{\tau\}$ when $\overline{T} = \overline{\{123, 132, 231\}}$ and, τ contains one permutation in T or $\tau = 3412$	282	n	theorem 4.3
$T \cup \{\tau\}$ when $123, 321 \in T$ or ($123 \in T$ and $\tau = 4321$) or ($321 \in T$ and $\tau = 1234$)	108	0	Erdös and Szekeres [3]
$\overline{\{123, 132, 231, \tau\}}$, $\tau = 3214$, 4312 or 4213 $\overline{\{123, 213, 231, \tau\}}$, $\tau = 1432$, 4132 or 4312 $\overline{\{123, 231, 312, \tau\}}$, $\tau = 1432$, 2143 or 3214 $\overline{\{132, 213, 231, \tau\}}$, $\tau = 1234$, 4123, 4321 or 4312	46	3	theorem 4.5
$T \cup \{\tau\}$ when $\overline{T} = \{123, 132, 213\}$ and, τ contains one permutation in T	38	f_{n+1}	theorem 4.1
$\overline{\{123, 231, 312, \tau\}}$, $\tau = 3421$ or 4231	6	4	theorem 4.4

Table 3: Cardinalities of the sets $S_n(T, \tau)$ when $T \subset S_3$, $|T| = 3$ and $\tau \in S_4$.

5 $|S_n(T, \tau)|$ when $T \subset S_3$, $|T| = 4, 5, 6$ and $\tau \in S_4$

In this section we calculate the cardinalities of all the sets $S_n(T, \tau)$ when $|T| \geq 4$, $T \subset S_3$ and τ is any permutation in S_4.

Theorem 5.1 For all $n \geq 4$, $S_n(123, 132, 213, 231, 4312) = S_n(123, 132, 231, 312, 3214) = S_n(132, 213, 231, 312, 1234) = \{(n, \ldots, 1)\}$.

Proof By induction and proposition 4.1 . ■

Theorem 5.2 Let $T \subset S_3$, $|T| = 4$, $\{123, 321\} \not\subset T$ and let $\tau \in S_4$ contains at least one permutation in T. Then $|S_n(T, \tau)| = 2$.

Proof By Simion and Schmidt [7] we have that $|S_n(T)| = 2$, so by theorem 2.2. ■

Theorem 5.3 Let $T \subset S_3$ and $|T| = 5$. Then for all $n \geq 3$,

$$|S_n(T, \tau)| = 1,$$

in the following cases:

1. $S_n(T, \tau) = \{(1, 2, \dots, n)\}$ if $321 \notin T$ and $\tau \neq 4321$.
2. $S_n(T, \tau) = \{(n, n-1, \dots, 1)\}$ if $123 \notin T$ and $\tau \neq 1234$.

Proof By induction and proposition 4.1 . ■

Representative $T \in \mathcal{T}$	$\lvert\mathcal{T}\rvert$	$\lvert S_n(T)\rvert$ *for* $T \in \mathcal{T}$	*Reference*
$T \cup \{\tau\}$, $\lvert T\rvert \geq 4$ when $123, 321 \in T$, or $123 \in T$ and $\tau = 4321$, or $321 \in T$ and $\tau = 1234$	348	0	Erdös and Szekeres [3]
$T \cup \{\tau\}$, $\lvert T\rvert = 4$, $\{123, 321\} \not\subset T$ and $\tau \in S_4$ contains permutation in T	100	2	theorem 5.2
$T \cup \{\tau\}$ when $\lvert T\rvert = 5$, $123 \notin T$ and $\tau \neq 1234$, or $\lvert T\rvert = 5$, $321 \notin T$ and $\tau \neq 4321$, or $\overline{\{123, 132, 213, 231, 4312\}}$, $\overline{\{123, 132, 231, 312, 3214\}}$, $\overline{\{123, 213, 231, 312, 1432\}}$, $\overline{\{132, 213, 231, 312, 1234\}}$,	56	1	theorem 5.1 5.3

Table 4: Cardinalities of the sets $S_n(T, \tau)$ when $T \subset S_3$, $|T| \geq 4$ and $\tau \in S_4$.

Theorem 5.4 Let $T \subset S_3$. Then for all $n \geq 6$,

$$|S_n(T, \tau)| = 0,$$

in the following cases:

1. $123 \in T$ and $\tau = 4321$.
2. $321 \in T$ and $\tau = 1234$.
3. $\{123, 321\} \subset T$ and $\tau \in S_4$.

Proof By Erdös and Szekeres [3]. ∎

All these results we summarize in the table [3].

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